Strongly Minimal Sets and the Polynomial Reducts of the Complex Field

by

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Dedication

To Yvonne.

Chapter 1

Introduction

The point of this paper is to offer a detailed exposition of one out of three cases from the recent paper of Castle and Tran [CT23] classifying polynomial reducts of the complex field. The complete result says, roughly, that "most" sets of (multivariate) complex polynomials are capable of defining $+$ and $\times$.

Theorem 1.1. *Suppose $\mathcal{P}$ is a collection of complex polynomial maps from various $\mathbb{C}^n$ to $\mathbb{C}$, and let $M = (\mathbb{C}, \mathcal{P})$ be the first order structure in the language where there is a function symbol for each member of $\mathcal{P}$. Then M is interdefinable with exactly one of the following:*

1) $(\mathbb{C}, \mathcal{U})$, *where $\mathcal{U}$ is a set of unary polynomials;*

2) $(\mathbb{C}, +, (\lambda \cdot)_{\lambda \in F})$, *the F-vector space structure, where F is a subfield of $\mathbb{C}$;*

3) $(\mathbb{C}, \times_r)$ *for some $r \in \mathbb{C}$, where $\times_r$ is the twisted multiplication $x \times_r y = (x - r)(y - r) + r$;*

4) $(\mathbb{C}, +, \times)$.

By "interdefinable" structures with the same universe we mean that the class of definable sets with parameters coincide.

The situation naturally divides into three cases based on the pregeometry on the strongly minimal structure $M = (\mathbb{C}, \mathcal{P})$ (see chapters 2 and 3 for detailed definitions).

The first case corresponds to the "trivial" case, as it occurs exactly when the pregeometry of M is trivial (see chapter 2). The union of the second and third cases corresponds to nontrivial locally modular pregeometries and is the focus of this essay. The last case, when the pregeometry of M is not locally modular, will not be discussed here; the proof in [CT23] that M is interdefinable with $(\mathbb{C}, +, \times)$ uses the Restricted Zilber Trichotomy on $\mathbb{C}$ from [Cas24].

Therefore, the main theorem we end up proving is the following.

Theorem 1.2. *Suppose $\mathcal{P}$ is a collection of complex polynomial maps, and let $M = (\mathbb{C}, \mathcal{P})$. If M has nontrivial, locally modular pregeometry, then M is interdefinable with $(\mathbb{C}, +, (\lambda\cdot)_{\lambda\in F})$ where F is a subfield of $\mathbb{C}$, or with $(\mathbb{C}, \times_r)$ for some $r \in \mathbb{C}$.*

While studying an earlier version of [CT23], I found a mistake in a combinatorial argument occuring in the proof of the nontrivial locally modular case of Theorem 1.1, which I communicated to the authors. They were able to find an alternative argument that avoids the error, and it is this new argument that I am giving an exposition of here. It constitutes the bulk of chapter 4.

I will assume the background of a standard graduate one-semester course on model theory, which covers topics such as types and saturation, as well as familiarity with more advanced concepts from stabilty theory such as Morley rank and canonical bases. As the main goal of this paper is to prove results from [CT23], the preliminary chapters often reference facts without proof. Some proofs are included either because they are highly beneficial for one's understanding or because good references cannot be easily found (folklore).

Chapter 2 introduces the key notions around pregeometries. The definitions are standard, but contains some proofs as some results required later cannot be found in the literature.

Chapter 3 explains how pregeometries arise in model theory. Some results regarding quotients are proved.

And finally, Chapter 4 establishes Theorem 1.2. The novel contributions of [CT23] all occur in this chapter. Therefore the readers familiar with the prerequisites may find it better to skip the previous chapters.

Chapter 2

Pregeometries

For a more detailed exposition of the theory of pregeometries, including proofs of the facts I will state, I suggest reading the appendix C of [TZ12].

As is common in model theory, juxtaposition of sets and elements denotes unions. For example, Aa means $A \cup \{a\}$.

Definition 2.1. A *pregeometry* on a set X is a closure operator $\mathrm{cl} : 2^X \to 2^X$ such that

- (Monotonicity) $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;
- (Finite character) $x \in \mathrm{cl}(A)$ implies $x \in \mathrm{cl}(A_0)$ for some finite $A_0 \subseteq A$;
- (Reflexivity) $A \subseteq \mathrm{cl}(A)$;
- (Transitivity) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;
- (Exchange) $b \in \mathrm{cl}(Aa) - \mathrm{cl}(A)$ implies $a \in \mathrm{cl}(Ab)$.

Usually, a pregeometry is introduced as a pair (X, cl). For this chapter, if X is not introduced explicitly, it denotes the pregeometry (X, cl).

The prototypical example of a pregeometry is a vector space with linear span as the closure operator. With pregeometries, we have an appropriate notion of independence.

Definition 2.2. A subset $A \subseteq X$ is called

1) *independent* if $a \notin \mathrm{cl}(A - \{a\})$ for all $a \in A$
2) a *generating set* if $\mathrm{cl}(A) = X$;
3) a *basis* if A is an independent generating set.

As expected, any independent set can always be extended to a basis, and all bases of a pregeometries have the same cardinality.

Fact 2.3. *Let $A \subseteq X$ be independent. Then, A can be extended to a basis of X.*

Fact 2.4. *Let $A, B \subseteq X$ be bases of X. Then, $|A| = |B|$.*

As we will see, this will allow us to define a notion of dimension for a pregeometry and all its subsets. But first, let us recall localization and restriction.

Definition 2.5 (Localization)**.** Let $A \subseteq X$. Define the operator cl_A on 2^X by $\mathrm{cl}_A(B) = \mathrm{cl}(A \cup B)$. Then $X_A := (X, \mathrm{cl}_A)$ is called the *localization of X at A*.

Fact 2.6. *If (X, cl) is a pregeometry, then (X, cl_A) is a pregeometry for any $A \subseteq X$.*

Definition 2.7 (Restriction)**.** Let $A \subseteq X$. Define the operator cl^A on 2^A by $\mathrm{cl}^A(B) = A \cap \mathrm{cl}(B)$. Then (A, cl^A) is called the *restriction of X to A*.

Fact 2.8. *If (X, cl) is a pregeometry, then (A, cl^A) is a pregeometry for any $A \subseteq X$.*

Using the fact that restrictions are pregeometries, we can define the dimension of a preogeometry as well as all its subsets at once.

Definition 2.9. The *dimension* of $A \subseteq X$, denoted $\dim(A)$, is defined as the cardinality of a basis of (A, cl^A).

Central to our discussion are the three properties of pregeometries: trivial, modular, and locally modular.

Definition 2.10. Suppose (X, cl) is a preogemetry.

1) (X, cl) is *trivial* if for any nonempty $A \subseteq X$, $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$.

2) (X, cl) is *modular* if for any closed sets $A, B \subseteq X$,

$$\dim(A \cup B) + \dim(A \cap B) = \dim(A) + \dim(B).$$

3) (X, cl) is *locally modular* there is some $e \in X$ such that the localization X_e of X at $\{e\}$ is modular.

Proposition 2.11. *The following are equivalent:*

1) (X, cl) is modular.

2) For all a, b, B ins X with $\dim(ab) = 2$, $\dim(ab/B) = 1$, there is $c \in \mathrm{cl}(B)$ such that $\dim(ab/c) = 1$.

As one would expect, dimension inside a pregeometry is entirely captured by its poset of closed sets.

Proposition 2.12. *Let (X, cl) and (X', cl') be pregeometries. Suppose there is surjection $f : X \to X'$ such that its induced map on 2^X commutes with closure (i.e $f \circ \mathrm{cl} = \mathrm{cl}' \circ f$) and respects $\subsetneq$ on closed sets (i.e if A, B are closed sets such that A is a proper subset of B, then $f(A)$ is a proper subset of $f(B)$). Then for any $A \subseteq X$, $\dim(A) = \dim'(f(A))$ where $\dim, \dim'$ are the dimension functions on X, X', respectively.*

Proof. Since the dimension of a set is equal to the dimension of its closure, it is enough to show $\dim(\mathrm{cl}(A)) = \dim'(\mathrm{cl}'(f(A)))$. But $\mathrm{cl}'(f(A)) = f(\mathrm{cl}(A))$. So we may assume A is closed.

First observe that the preimage of a closed set is closed. Indeed, for any closed $C' \subseteq X'$, set $C = f^{-1}(C')$. Then $f(\mathrm{cl}(C)) = \mathrm{cl}'(f(C)) = \mathrm{cl}'(C') = C'$, where the first equality uses the fact that F commutes with closure, while the second equality uses that f is surjective. Thus $\mathrm{cl}(C) \subseteq f^{-1}(C') = C$, which means C is closed.

Also, from the fact that f commutes with closure, it follows immediately that f takes closed sets to closed sets.

Next, observe that the dimension of a set is equal to the dimension of its closure. So to show $\dim(A) = \dim'(f(A))$, it is enough to show $\dim(\mathrm{cl}(A)) = \dim'(\mathrm{cl}'(f(A))) = \dim'(f(\mathrm{cl}(A)))$. Thus, we may assume A is closed.

It is easy to check that the dimension of A is the largest cardinal κ such that there is a nested chain of closed sets $A_0 \subsetneq \ldots \subsetneq A_\kappa$ with $A_0 = \mathrm{cl}(\varnothing)$ and $A_\kappa = A$. Apply f to this chain. As f respects $\subsetneq$, we get a chain of increasing closed sets (note that f must take closed sets to closed sets) that unions to $f(A)$. Therefore, $\dim'(f(A)) \geq \dim(A)$.

Similarly, we can take the preimage of a chain to see $\dim'(f(A)) \leq \dim(A)$. □

Proposition 2.13. *Let (X, cl) be a pregeometry and let $\sim$ be an equivalence relation on X such that closed sets are unions of $\sim$-classes. Then, $(X/\sim, \mathrm{cl}_\sim)$ is a pregeometry, where the closure operator $\mathrm{cl}_\sim$ is given by*

$$\mathrm{cl}_\sim(A/\sim) := \mathrm{cl}(A)/\sim .$$

Moreover, $\pi : X \to X/\sim$ satisfies the conditions of 2.12.

Proof. We check that $\mathrm{cl}_\sim$ is well-defined: if $A/\sim = B/\sim$, then $\mathrm{cl}(A) = \mathrm{cl}(B)$ because closed sets are unions of $\sim$-classes; thus $\mathrm{cl}(A)/\sim = \mathrm{cl}(B)/\sim$.

All conditions of pregeometries except for exchange are easy to check. For exchange, notice that

$$b/\sim \in \mathrm{cl}_\sim(A/\sim) \iff b \in \mathrm{cl}(A)/\sim \iff b \in \mathrm{cl}(A).$$

Therefore it follows that

$$\begin{aligned} b/\sim \in \mathrm{cl}_\sim(Aa/\sim) - \mathrm{cl}_\sim(A/\sim) &\iff b/\sim \in \mathrm{cl}_\sim(Aa/\sim) \text{ and } b/\sim \notin \mathrm{cl}_\sim(A/\sim) \\ &\iff b \in \mathrm{cl}(Aa) \text{ and } b \notin \mathrm{cl}(A) \\ &\implies a \in \mathrm{cl}(Ab) \\ &\iff a/\sim \in \mathrm{cl}_\sim(Ab/\sim). \end{aligned}$$

It remains to check conditions of 2.12. By definition, $\mathrm{cl}_\sim$ commutes with the projection, which is also clearly surjective. Finally, let $A \subsetneq B$ be closed. Then, there is some $b \in B - A$, which means $b/\sim \in (B/\sim) - (A/\sim)$. □

Corollary 2.14. *Let (X, cl) be a pregeometry and let $\sim$ be an equivalence relation on X such that closed sets are unions of $\sim$-classes.*

1) X is trivial if and only if $X/\sim$ is trivial.

2) X is modular if and only if $X/\sim$ is modular.

Proof. 1) This follows from the observation that $b/\sim \in \mathrm{cl}_\sim(A/\sim)$ if and only if $b \in \mathrm{cl}(A)$.

2) Suppose X is modular. Let $\pi : X \to X/\sim$ be the natural quotient. Let $A', B' \subseteq X/\sim$ be closed. Then, $A = \pi^{-1}(A')$ and $B = \pi^{-1}(B')$ are closed. Note that $\cap$ and $\cup$ are compatible with π. Applying Proposition 2.13 to

$$\dim(A \cup B) + \dim(A \cap B) = \dim(A) + \dim(B),$$

we get

$$\dim_\sim(A' \cup B') + \dim_\sim(A' \cap B') = \dim_\sim(A') + \dim_\sim(B').$$

The other direction is analogous.

□

Chapter 3

Pregeometries, Now with Logic

One way in which pregeometries arise in model theory is from strongly minimal sets. We refer the reader to [TZ12] for details on the model-theoretic facts that we use throughout this chapter.

We fix a (possibly multi-sorted) complete, stable L-theory T eliminating imaginaries. We work in a fixed universal domain $\mathbb{M}$ for T. Specifically, for some sufficiently large cardinal κ, $\mathbb{M} \models T$ is a κ-saturated, strongly κ-homogeneous model (the results in this section are also valid for class-sized models). All parameter sets $A, B, \ldots$ are assumed to be in $\mathbb{M}$ and small, namely of cardinality $< \kappa$.

Recall that a definable set D is *strongly minimal* if every definable subset is either finite or cofinite.

Fact 3.1 (D_A is a pregeometry, 5.7.5 of [TZ12])**.** *Let D be an A-definable strongly minimal set. Let cl_A be the closure operator on 2^D given by $\mathrm{cl}_A(X) = \mathrm{acl}(X \cup A) \cap D$. Then, $D_A := (D, \mathrm{cl}_A)$ is a pregeometry.*

Notice the resemblance of this notation to localization in the previous chapter. The dimension function of D_A will be denoted by $\dim_A$ or sometimes by $\dim(\cdot/A)$.

Fact 3.2 ($\dim$ = RM in strongly minimal sets, 6.4.2 of [TZ12])**.** *Let D be an A-definable strongly minimal set. Suppose $a_1, \ldots, a_n \in D$. Then $\dim(a_1, \ldots, a_n/A) = \mathrm{RM}(a_1, \ldots, a_n/A)$.*

Fact 3.3 (additivity of RM in $\mathrm{acl}(D)$, 6.4.9 of [TZ12])**.** *Let D be a $\varnothing$-definable strongly minimal set and let $\overline{a}, \overline{b} \in \mathrm{acl}(D)$ be tuples (of possibly various sorts). Then for any B,*

$$\mathrm{RM}(\overline{a}, \overline{b}/B) = \mathrm{RM}(\overline{a}/B) + \mathrm{RM}(\overline{b}/B\overline{a}).$$

Definition 3.4. For a $\varnothing$-definable strongly minimal set D, $D^{eq} := \mathrm{dcl}(D)$.

It is not hard to see that D^{eq} is the union of the images of all D-definable functions from subsets of D^n to M. Equivalently, it contains the code of every D-definable subset of D^n.

We will be using some more advanced notion from geometric stability theory. In particular, that of a nonforking extension of a complete type. A type $p \in S(A)$ is stationary if it has a unique nonforking extension to $S(B)$ for every $B \supseteq A$. Every type over an algebraic closed set is stationary (recall that T eliminates imaginary, see 8.5 of [TZ12]). We therefore often consider the strong type of a over A, namely $\mathrm{stp}(a/A) := \mathrm{tp}(a/\operatorname{acl}(A))$.

Given a stationary type, $p \in S(A)$, we can consider the unique global nonforking extension of p to $S(\mathbb{M})$, say $\mathfrak{p}$. By a canonical base of p, we mean a set $C \subseteq \operatorname{dcl}(A)$ such that for all $\sigma \in \mathrm{Aut}(\mathbb{M})$, $\sigma(\mathfrak{p}) = \mathfrak{p}$ if and only if σ is identity on C. Canonical bases always exist.

Given two strongly minimal sets X and Y, we say X *almost equals* (a.e) Y if they have finite symmetric difference.

Proposition 3.5. *Let D be a $\varnothing$-definable strongly minimal set, $\overline{a} = (a_1, \ldots, a_n) \in D^n$ a finite tuple, and B be a small set. Then, there is a finite tuple $\overline{c} \in D^{eq}$ such that $\overline{c}$ is canonical base of $\mathrm{stp}(\overline{a}/B)$.*

Proof. We have the following key observation that holds for any global type $\mathfrak{p}(\overline{x}) \in S(\mathbb{M})$ with Morley rank k and $\psi(\overline{x}, \overline{a}) \in \mathfrak{p}(\overline{x})$ of Morley rank k and degree 1: for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$, whether $\sigma(\mathfrak{p}) = \mathfrak{p}$ depends exactly on whether $\psi(\mathbb{M}, \sigma(\overline{d}))$ almost equals $\psi(\mathbb{M}, \overline{d})$. It is clear that $\sigma(\mathfrak{p}) = \mathfrak{p}$ implies almost equality. To see the other direction, notice that $\sigma(\mathfrak{p})$ is a global type with Morley rank k and degree 1 as well, and $\psi(\overline{x}, \sigma(\overline{d}))$ is a formula with Morley rank k and degree 1 in $\sigma(\mathfrak{p})$. For each degree 1 formula, there is exactly one type with the same rank containing the formula. In this case, $\mathfrak{p}$ is the unique global type containing $\psi(\overline{x}, \overline{d})$ with the same rank, and similarly for $\sigma(\mathfrak{p})$ and $\psi(\overline{x}, \sigma(\overline{d}))$. But if $\psi(\mathbb{M}, \sigma(\overline{d}))$ almost equals $\psi(\mathbb{M}, \overline{d})$, then both $\mathfrak{p}$ and $\sigma(\mathfrak{p})$ are global types containing $\psi(\overline{x}, \overline{d}) \wedge \psi(\overline{x}, \sigma(\overline{d}))$. By uniqueness, we must have $\mathfrak{p} = \sigma(\mathfrak{p})$.

Suppose $\dim(a_1, \ldots, a_n/B) = k$. Rearrange so that $a_1, \ldots, a_k$ are independent over B. Let $\phi(x_1, \ldots, x_n)$ be an $L_{\mathrm{acl}(B)}$-formula in $\mathrm{stp}(\overline{a}/B)$ with minimum Morley rank and degree. We may assume $\phi(\mathbb{M}) \subseteq D^n$. Because strong types are stationary, $\mathrm{dM}(\phi) = 1$. Let $\mathfrak{p}(\overline{x}) \in S(\mathbb{M})$ be the unique global nonforking extension of $\mathrm{stp}(\overline{a}/B)$.

In a stable theory, every formula is stably embedded, so we can find an L-formula $\psi(x_1, \ldots, x_n, \overline{y})$ and parameters $\overline{d} \in D$ such that $\psi(x_1, \ldots, x_n, \overline{d})$ defines the same set as $\phi(x_1, \ldots, x_n)$, and $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ implies $x_i \in D$ for all i.

From our observation above, we see that an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$ fixes $\mathfrak{p}$ if and only if $\psi(\mathbb{M}, \sigma(\overline{d}))$ almost equals $\psi(\mathbb{M}, \overline{d})$.

Because RM is definable in strongly minimal sets, we see that

$$X = \{\overline{e} \in D : \mathrm{RM}(\psi(\mathbb{M}, \overline{e}) \cap \psi(\mathbb{M}, \overline{d})) = k\}$$

is $\overline{d}$-definable.

For any automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$, $\sigma(X) = X$ if and only if $\psi(\mathbb{M}, \overline{d})$ almost equals $\psi(\mathbb{M}, \sigma(\overline{d}))$. Indeed, the forward direction is direct, and the backward direction

is as follows. Suppose $\psi(\mathbb{M}, \overline{d})$ almost equals $\psi(\mathbb{M}, \sigma(\overline{d}))$, by which it follows

$$\begin{aligned}\overline{e} \in X &\iff \mathrm{RM}(\psi(\mathbb{M}, \overline{e}) \cap \psi(\mathbb{M}, \overline{d})) = k \\ &\iff \mathrm{RM}(\psi(\mathbb{M}, \overline{e}) \cap \psi(\mathbb{M}, \sigma(\overline{d}))) = k \\ &\iff \overline{e} \in \sigma(X).\end{aligned}$$

Let $c \in D^{eq}$ be a code of X. We claim that it is a canonical base of $\mathrm{stp}(\overline{a}/B)$. Indeed, for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$,

$$\begin{aligned}\sigma(c) = \sigma(c) &\iff \sigma(X) = X \\ &\iff \psi(\mathbb{M}, \overline{d}) \text{ almost equals } \psi(\mathbb{M}, \sigma(\overline{d})) \\ &\iff \sigma(\mathbb{p}) = \mathbb{p}.\end{aligned}$$

□

For the following discussion D is an A-definable strongly minimal set.

It turns out that local modularity, a combinatorial condition on D_A, has a lot to do with dimensions of definable families of plane curves in D. Here, by a plane curve we mean a definable subset of D^2 of Morley rank 1 and degree 1 (i.e strongly minimal).

It makes sense to say a single curve is definable, but what does it mean for a family of plane curves to be definable?

For a family to have a suitable notion of definability, it makes sense to require that there exists a fixed formula $\phi(x_1, x_2, \overline{y})$ such that the family consists of curves defined by $\phi(x_1, x_2, \overline{a}_i)$ where the parameters $\overline{a}_i$ are allowed to vary in the family. Definability of the family then refers to the definability of $\{\overline{a}_i : i \in I\}$.

Unfortunately, because we are fixing a formula ϕ, it is possible that some $\phi(D^2, \overline{a}_i)$ is no longer a curve. For example, consider the family $C_{a,b} = \{(x, y) \in \mathbb{C}^2 : ax + by = 0\}$ where $a, b \in \mathbb{C}$ definable in $(\mathbb{C}, +, \times)$. Take $a = 0, b = 0$, then $C_{a,b}$ is just $\mathbb{C}^2$. The solution is to require that the parameters have the same type. Indeed, because Morley rank and degree of $\phi(x_1, x_2, \overline{a}_i)$ only depend on the type of $\overline{a}_i$, this ensures every set in the family is a curve. This leads us to the following definition.

Definition 3.6. Let D be an A-definable strongly minimal set. A *definable family of plane curves* in D_A is an L_A-formula $\phi(x_1, x_2, \overline{y})$ together with a tuple $\overline{a} \in \mathbb{M}$ such that $\phi(\mathbb{M}, \overline{a})$ is a strongly minimal set in D^2.

Intuitively, this family consists of all curves defined by $\phi(x_1, x_2, \overline{a}')$ where $\overline{a}'$ has the same type as $\overline{a}$ over A. For convenience, we usually write the family as $\phi(x_1, x_2, \overline{a})$ (this abuse of notation is intentional: we identify each curve with the family of curves it belongs to).

Naively, we can measure the dimension of this family by measuring $\mathrm{RM}(\overline{a}/A)$. However, this dimension can be arbitrarily large and curves may "overlap". Consider

the case where $\overline{y}$ actually does not occur in ϕ and $a_1, \ldots, a_n$ are algebraically independent over A, then the proposed dimension of the family $(\phi, \overline{a})$ is n even though there is only really one curve in this family. Therefore, a desirable notion of dimension should account for curves that are almost equal.

Given a family $(\phi, \overline{a})$ of curves, let $\mathcal{F}$ denote the set-theoretic family

$$\{\phi(\mathbb{M}, \overline{a}') : \overline{a}' \models \mathrm{tp}(\overline{a}/A)\}.$$

Let $\sim$ be the equivalence relation on $\mathcal{F}$ given by almost equivalence. We are in fact interested in the "dimension" of $\mathcal{F}/\sim$. This is precisely where canonical bases are useful.

Let $X = \phi(\mathbb{M}, \overline{a})$ be a strongly minimal set in D^2, where ϕ can have parameters from A. Then, there is a unique global generic type $\mathbb{p}(\overline{x}) \in S(\mathbb{M})$ of $\phi(\overline{x}, \overline{a})$). For any automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, $\sigma(\mathbb{p}) = \mathbb{p}$ if and only if $\sigma(X) = \phi(\mathbb{M}, \sigma(\overline{a}))$ almost equals X (see comments on this in the proof of Proposition 3.5). On the other hand, we know that $\sigma(\mathbb{p}) = \mathbb{p}$ if and only if $\sigma(c) = c$ where c is a canonical base of $\mathbb{p}$. Therefore, it makes sense to think of $\mathrm{RM}(c/A)$ as the dimension of $\mathcal{F}/\sim$. This inspires the following definition.

Definition 3.7. Let D be an A-definable strongly minimal set. Then, the *dimension* of a family of plane curves $\phi(\mathbb{M}, \overline{a})$ in D_A is defined as $\mathrm{RM}(c/A)$ where $c \in \mathbb{M}$ is a canonical base of the global generic type of $\phi(\mathbb{M}, \overline{a})$.

Definition 3.8. Let D be an A-definable strongly minimal set. Then D_A is *linear* if every family of plane curves in D_A has dimension at most 1.

Observe that for any $a_1, a_2 \in D$ and $B \supseteq A$, if $\mathrm{RM}(a_1, a_2/B) = 1$, then $\mathrm{stp}(a_1, a_2/B)$ is the generic type of some curve containing a_1, a_2. Therefore, the linearity of D_A amounts to saying: for all $a_1, a_2 \in D$ and $B \supseteq A$, if $\mathrm{RM}(a_1, a_2/B) = 1$ and c is a canonical base of $\mathrm{stp}(a_1, a_2/B)$, then $\mathrm{RM}(c/A) \leq 1$.

Because we are working in a stable theory eliminating imaginaries, every stationary type with Morley rank has a finite canonical base c. Together with the fact that any two canonical bases of the same type are definably equivalent (hence have the same RM over A), linearity of D_A is the same as requiring the *existence* of a canonical base c such that $\mathrm{RM}(c/A) \leq 1$.

Theorem 3.9. *Let D be a strongly minimal set in $\mathbb{M}$ definable over A. Then, the following are equivalent.*

1. *For some small $B \supseteq A$, D_B is modular.*
2. *D_A is linear.*
3. *D_A is locally modular.*

Proof. We begin by showing that we can reduce to the case where $A = \varnothing$. Let L' be L with A added as constant symbols. Note that $\mathbb{M}_A$ is a saturated model of $T' = \mathrm{Th}^{L'}(\mathbb{M}_A)$, which is still complete and stable, and eliminates imaginaries. Moreover, D is now $\varnothing$-definable. For clarity, let D' denote D as a $\varnothing$-definable set in $\mathbb{M}_A$. Consider the following conditions.

1′. For some small B, D'_B is modular.

2′. D' is linear.

3′. D' is locally modular.

It is clear that condition 3 is equivalent to 3′ because D' has the same preogeometry as D_A and local modularity concerns only the progeometry structure.

Suppose condition 1 holds. Let $B \supseteq A$ be a small set such that D_B is modular. Then, D'_B has the same pregeometry as D_B. So D'_B is modular as $B \supseteq A$. Conversely, say 1′ holds. Let B' be a small set such that D'_B is modular. Then, D_B is modular where $B = A \cup B'$.

To justify the reduction, it remains to show 2 is equivalent to 2′. Note that for any tuple a and parameters E, $\mathrm{RM}^{L'}(a/E) = \mathrm{RM}^{L}(a/A \cup E)$, and if $\mathrm{RM}(a/E)$ is bounded, then $\varnothing \neq \mathrm{Cb}^{L}(\mathrm{stp}^{L}(a/A \cup E)) \subseteq \mathrm{Cb}^{L'}(\mathrm{stp}^{L'}(a/E))$. The nonemptiness in the latter is because stationary types with bounded Morley ranks have finite canonical bases in saturated models of stable theories. We therefore have:

$$
\begin{aligned}
& D_A \text{ is linear} \\
\iff & \begin{bmatrix} \forall E \supseteq A\ \forall a_1, a_2 \in D \text{ s.t } \mathrm{RM}^{L}(a_1, a_2/E) = 1, \\ \exists c \in \mathbb{M}, (c = \mathrm{Cb}^{L}(\mathrm{stp}^{L}(a_1, a_2/E)) \wedge \mathrm{RM}^{L}(c/A) \leq 1) \end{bmatrix} \\
\iff & \begin{bmatrix} \forall E\ \forall a_1, a_2 \in D \text{ s.t } \mathrm{RM}^{L}(a_1, a_2/A \cup E) = 1, \\ \exists c \in \mathbb{M}, (c = \mathrm{Cb}^{L}(\mathrm{stp}^{L}(a_1, a_2/A \cup E)) \wedge \mathrm{RM}^{L}(c/A) \leq 1) \end{bmatrix} \\
\iff & \begin{bmatrix} \forall E\ \forall a_1, a_2 \in D' \text{ s.t } \mathrm{RM}^{L'}(a_1, a_2/E) = 1, \\ \exists c \in \mathbb{M}, (c = \mathrm{Cb}^{L'}(\mathrm{stp}^{L'}(a_1, a_2/E)) \wedge \mathrm{RM}^{L'}(c) \leq 1) \end{bmatrix} \\
\iff & D' \text{ is linear.}
\end{aligned}
$$

This completes the reduction and we may assume $A = \varnothing$.

$(1 \implies 2)$. Let B be such that D_B is modular. Let $a_1, a_2 \in D$ and E be such that $\mathrm{RM}(a_1, a_2/E) = 1$. Let $c \in \mathbb{M}$ be a canonical base of $\mathrm{stp}(a_1, a_2/E)$. By Proposition 3.5, we may assume $c \in \mathrm{dcl}(D)$. We want to show $\mathrm{RM}(c) \leq 1$.

First a quick observation: if $\mathrm{RM}(a_1, a_2) \leq 1$, then $\mathrm{tp}(a_1, a_2/E)$ does not fork over $\varnothing$; so $c \in \mathrm{acl}(\varnothing)$, giving $\mathrm{RM}(c) = 0$. Let us thus suppose $\mathrm{RM}(a_1, a_2) = 2$.

Note that our goal only depends on the global nonforking extension $\mathbb{p}(x_1, x_2) \in S(M)$ of $\mathrm{stp}(a_1, a_2/E)$ because the canonical base of a stationary type is defined to be the canonical base of its global nonforking extension. So we are free to adjust a_1, a_2, E as long as the nonforking extension of $\mathrm{stp}(a_1, a_2/E)$ remains $\mathbb{p}$. Because

Morley rank of types are defined as the minimal rank of formulas in them, there is $\phi(x_1, x_2, \overline{e}) \in \mathrm{stp}(a_1, a_2/E)$ defining a strongly minimal subset $X \subseteq D^2$. Then, because formulas in stable theories are stably embedded, X is defined over some tuple $\overline{e}'$ from D, say by $\phi'(x_1, x_2, \overline{e}')$. As $\phi'(x_1, x_2, \overline{e}')$ is equivalent to $\phi(x_1, x_2, \overline{e})$, we have $\phi'(x_1, x_2, \overline{e}') \in \mathbb{p}$. Let q be $p \restriction \mathrm{acl}(\overline{e}')$. By saturation, q is realized by some $a_1', a_2' \in D$. As $\mathrm{RM}(\mathbb{p}) = 1 = \mathrm{RM}(q)$, $\mathbb{p}$ is the gloabl nonforking extension of q. Therefore, , we may assume E is a finite tuple in D by replacing a_1, a_2, E with $a_1', a_2', \overline{e'}$.

If $\sigma \in \mathrm{Aut}(\mathbb{M})$, then $\sigma(c)$ is a canonical base of $\sigma(\mathbb{p})$ and $\mathrm{RM}(\sigma(c)) = \mathrm{RM}(c)$. This allows us to further simplify to the case where $a_1a_2E \mathop{\smile\!\!\!\!\!\!|} B$. Indeed, there is a nonforking extension of $\mathrm{tp}(a_1a_2E)$ to B. By saturation, this is realized by some $a_1'a_2'E'$. In particular, we have $\mathrm{tp}(a_1a_2E) = \mathrm{tp}(a_1'a_2'E')$. Because saturation implies strong homogeneity there is $\sigma \in \mathrm{Aut}(\mathbb{M})$ sending a_1a_2E to $a_1'a_2'E'$. Replacing a_1a_2E by $a_1'a_2'E'$, we may assume $a_1a_2E \mathop{\smile\!\!\!\!\!\!|} B$.

To recap, we have reduced to the case where $\mathrm{RM}(a_1, a_2) = 2$, E is a finite subset of D, and $a_1a_2E \mathop{\smile\!\!\!\!\!\!|} B$. Our goal is to show $\mathrm{RM}(c) \leq 1$ where $c \in \mathrm{dcl}(D)$ is a canonical base of $\mathrm{stp}(a_1, a_2/E)$.

Here are some direct observations:

- $c \in \mathrm{acl}(E)$ (3.0.1)
- $\mathrm{RM}(a_1, a_2/B) = 2$ because $\mathrm{RM}(a_1, a_2) = 2$ and $a_1a_2 \mathop{\smile\!\!\!\!\!\!|} B$; (3.0.2)
- $a_1a_2 \mathop{\smile\!\!\!\!\!\!|}_E B$ because $a_1a_2E \mathop{\smile\!\!\!\!\!\!|} B$; (3.0.3)
- $\mathrm{RM}(a_1, a_2/B \cup E) = \mathrm{RM}(a_1, a_2/E) = 1$ by 3.0.3; (3.0.4)
- c is also a canonical base of $\mathrm{stp}(a_1, a_2/E \cup B)$ by 3.0.3. (3.0.5)

Applying modularity of D_B to $\mathrm{cl}_B(a_1, a_2)$ and $\mathrm{cl}_B(E)$, we get

$$\begin{aligned}\dim_B(\mathrm{cl}_B(a_1, a_2) \cap \mathrm{cl}_B(E)) &= \dim_B(a_1, a_2) + \dim_B(E) - \dim_B(a_1a_2E) \\ &= \dim_B(a_1, a_2) + \dim_B(E) - (\dim_B(a_1, a_2/E) + \dim_B(E)) \\ &= 2 - 1 = 1,\end{aligned}$$

where the second equality follows from 3.0.2, 3.3, and 3.0.4.

Therefore, there exists $d \in D$ such that $d \in \mathrm{cl}_B(a_1, a_2) \cap \mathrm{cl}_B(E)$ and $d \notin \mathrm{cl}_B(\varnothing)$. It must be the case that $\dim_B(a_1, a_2/d) = 1$. Indeed, if $\dim_B(a_1, a_2/d) = 2$, then a_1, a_2, d are independent over B, contradicting $d \in \mathrm{cl}_B(a_1, a_2)$. Therefore, $\mathrm{stp}(a_1, a_1/E \cup B)$ does not fork over Bd. So $c \in \mathrm{acl}(Bd)$. But $\dim_B(d) = 1$, so $\mathrm{RM}(c/B) \leq 1$. From $E \mathop{\smile\!\!\!\!\!\!|} B$ and $c \in \mathrm{acl}(E)$, we get $c \mathop{\smile\!\!\!\!\!\!|} B$ as follows. This contradicts $E \mathop{\smile\!\!\!\!\!\!|} B$. Hence $c \mathop{\smile\!\!\!\!\!\!|} B$, whereby $\mathrm{RM}(c) = \mathrm{RM}(c/B) \leq 1$, as desired.

$(2 \Longrightarrow 3)$. Assume D is linear. Let $e \in D - \mathrm{acl}(\varnothing)$, we show D_e is modular. By 2.11, it is enough to show: for any $a_1, a_2 \in D$, $B \subseteq D$ finite, if $\mathrm{RM}(a_1, a_2/e) = 2$ and $\mathrm{RM}(a_1, a_2/Be) = 1$, then there exists $d \in \mathrm{cl}_e(B)$ such that $\dim_e(a_1, a_2/d) = 1$.

Let c be a canonical base of $\mathrm{stp}(a_1, a_2/Be)$. Then $c \in \mathrm{acl}(Be)$. By linearity, we know $\mathrm{RM}(c) \leq 1$. However, if $\mathrm{RM}(c) = 0$, then $\mathrm{RM}(a_1, a_2/e) \leq \mathrm{RM}(a_1, a_2) = \mathrm{RM}(a_1, a_2/c) = \mathrm{RM}(a_1, a_2/Be) = 1$, which is a contradiction. So $\mathrm{RM}(c)$ has to be 1.

By 3.3, we can write

$$\mathrm{RM}(a_1, a_2, c) = \mathrm{RM}(a_1, a_2/c) + \mathrm{RM}(c) = \mathrm{RM}(c/a_1, a_2) + \mathrm{RM}(a_1, a_2).$$

Because $\mathrm{RM}(a_1, a_2/c) = 1$, $\mathrm{RM}(c) = 1$, and $\mathrm{RM}(a_1, a_2) = 2$, we have $\mathrm{RM}(c/a_1, a_2) = 0$. So $c \in \mathrm{acl}(a_1, a_2)$.

If $a_1 \not\downarrow c$, then $a_1 \in \mathrm{acl}(c) \cap D \subseteq \mathrm{cl}_e(B)$ as $c \in \mathrm{acl}(Be)$ and we may take $d = a_1$ to complete the proof. Similarly for a_2. It remains to consider when $a_1 \downarrow c$ and $a_2 \downarrow c$. Because $a_1 a_2 \downarrow e$ and $c \in \mathrm{acl}(a_1, a_2)$, we get $c \downarrow e$. By forking symmetry, we get $e \downarrow c$, whereby $\mathrm{tp}(e/c) = \mathrm{tp}(a_1/c)$. By saturation, we can find $d \in D$ so that $\mathrm{tp}(a_1, a_2/c) = \mathrm{tp}(e, d/c)$. Then, because $a_2 \in \mathrm{acl}(a_1, c) \cap D$, we have $d \in \mathrm{acl}(e, c) \cap D$. Recall that $c \in \mathrm{acl}(Be) \cap \mathrm{acl}(a_1, a_2)$. Therefore, $d \in \mathrm{cl}_e(B) \cap \mathrm{cl}_e(a_1, a_2)$. If $\dim_e(a_1, a_2/d) = 2$, then $\dim_e(a_1, a_2, d) = 3$, contradicting $d \in \dim_e(a_1, a_2)$. Therefore, $\dim_e(a_1, a_2/d) = 1$ as desired.

$(3 \Longrightarrow 1)$. This is direct. □

Another reason Theorem 3.9 is interesting is that it implies that the local modularity of D_A does not depend on the choice of A.

Corollary 3.10. *Suppose the strongly minimal set D is A-definable and $A' \supseteq A$. Then D_A is locally modular if and only if $D_{A'}$ is locally modular.*

Proof. Suppose D_A is locally modular. So, by Theorem 3.9, D_A is linear. Let $E \supseteq A'$ and $a_1, a_2 \in D$ such that $\mathrm{RM}(a_1, a_2/E) = 1$. Then, by linearity of D_A, we know there is a canonical base c of $\mathrm{stp}(a_1, a_2/E)$ with $\mathrm{RM}(c/A) \leq 1$. But $\mathrm{RM}(c/A') \leq \mathrm{RM}(c/A)$. We have thus shown that $D_{A'}$ is also linear. Hence by 3.9, $D_{A'}$ is locally modular.

Suppose $D_{A'}$ is locally modular. Then, by Theorem 3.9, there is some B containing A' such that D_B is modular. By the theorem again, D_A is locally modular. □

Proposition 3.11. *Suppose the strongly minimal set D is A-definable and A'-definable. Then D_A is locally modular if and only if $D_{A'}$ is locally modular.*

Proof.

$$\begin{aligned} D_A \text{ is locally modular} &\Longleftrightarrow D_{A \cup A'} \text{ is locally modular} \\ &\Longleftrightarrow D_{A'} \text{ is locally modular.} \end{aligned}$$

□

From this, it is clear that we can talk about D being locally modular without referring to the parameters.

We mention some easy consequences.

Proposition 3.12. *Suppose D is a strongly minimal set, and $\sim$ is a definable equivalence relation on D with finite classes. Then D is locally modular if and only if $D/\sim$ is locally modular.*

Proof. Let A be such that D is A-definable and $\sim$ is A-definable. Then the closed sets in D_A are unions of $\sim$-classes.

By Proposition 3.11, it is enough to show that D_A is locally modular if and only if $(D/\sim)_A$ is locally modular. By Theorem 3.9 and possibly adding more elements to A, we can further reduce to showing D_A is modular if and only if $(D/\sim)_A$ is modular.

Directly apply Corollary 2.14 to complete the proof. □

Proposition 3.13. *Suppose strongly minimal sets C and D are in definable bijection. Then C is locally modular if and only if D is locally modular.*

Proof. Let $f : C \to D$ be the definable bijection. Let A be a small set so that all of C, D, f are A-definable. Let cl be the closure operator of C_A and cl' of D_A. Observe that C_A and D_A are "isomorphic pregeometries": for any $x \in C, X \subseteq C$, $x \in \mathrm{cl}(X)$ if and only $f(x) \in \mathrm{cl}'(f(X))$. Clearly local modularity is invariant under such isomorphisms. □

Proposition 3.14. *Let C, D be strongly minimal sets. Suppose $f : C \to D$ is a finite-to-one definable function with cofinite image. Then C is locally modular if and only if D is locally modular.*

Proof. By 3.12, we see that C is locally modular if and only if $C/\sim$ is locally modular where $\sim$ is the equivalence relation induced by f. By 3.13, $C/\sim$ is locally modular if and only if $f(C)$ is locally modular. We have thus reduced to the case where C is a cofinite subset of D and f is just the inclusion.

Let $A \supseteq D - C$. We show that D_A is modular if and only if C_A is modular. Recall that modular means for any X, Y closed, the following holds:

$$\dim(X \cup Y) + \dim(X \cap Y) = \dim(X) + \dim(Y). \tag{3.0.6}$$

Let $X_0, Y_0 \subseteq C$ be closed in C_A, and let X, Y be their closures in D_A respectively. We make the following observations:

- D_A and C_A agree on the dimension of any subset of C;
- taking the closure of a set in D_A is the same as first taking the closure of it in C_A and adding $D - C$ to it;
- the dimension of X_0 (resp. Y_0) is the same as the dimension of X (resp. Y) in D_A;
- the dimension of $X_0 \cup Y_0$ is the same as the dimension of $X \cup Y$ in D_A;

- for any set $X \subseteq D$, adding or removing $D - C$ from it does not change its dimension in D_A
- the dimension of a set $Z \subseteq D_A$ is the same as the dimension of $Z \cap C$;
- the dimension of $X_0 \cap Y_0$ is the same as the dimension of $X \cap Y$ in D_A.

Therefore, equation 3.0.6 holds for X_0, Y_0 if and only if it holds for X, Y. Together with the fact that any X, Y closed in D_A must come from taking the closure of some X_0, Y_0, we see that C_A is modular if and only if D_A is modular. □

Definition 3.15. A *definable finite correspondence* (or finite correspondence for short) between strongly minimal sets X and Y is a definable $C \subseteq X \times Y$ so that the projection from C to both X and Y are finite-to-one with cofinite image.

Corollary 3.16. *Local modularity is preserved by finite correspondence.*

Proof. Let $C \subseteq X \times Y$ be a finite correspondence between X and Y. Then, the projection $C \to X$ and $C \to Y$ are finite-to-one with cofinite image. By Propsition 3.14, we get that C is locally modular if and only if X is locally modular if and only if Y is. □

Chapter 4

Quasi-functions and Nontrivial Locally Modular Reducts

4.1 The Strategy

Our goal is to show that if $\mathcal{P}$ is a collection of complex polynomials such that $M = (\mathbb{C}, \mathcal{P})$ is a nontrivial and locally modular, then M is interdefinable with $(\mathbb{C}, +, (\lambda\cdot)_{\lambda \in F})$ for some subfield F of $\mathbb{C}$ or $(\mathbb{C}, \times_r)$ for some fixed r. Note that as $(\mathbb{C}, +, \times)$ is a strongly minimal structure, so is M. To clarify, it is the pregeometry on $\mathbb{C}$ induced by acl in M that we are assuming is nontrivial and locally modular.

From the trivial case of Theorem 1.1 (which we are assuming), we know the nontriviality of reduct M implies the existence of a binary polynomial in $\mathcal{P}$. Using the classical result of Hrushovski and Pillay on classification of definable sets in locally modular groups (introduced in 4.17), we may focus on binary polynomials definable in M.

We develop the notion of quasi-functions, a generalization of binary functions that is robust against small changes in the domain and codomain and against the act of passing through finite-to-one definable functions. On top of this, we introduce the notion of quasi-functions having *small symmetric expansion*, which is also preserved under the two kinds of aforementioned actions.

Intuitively, having small symmetric expansion means a quasi-function (think just a binary function) produces few outputs on $A \times A$ for increasingly large A in the domain. In the recent paper [JRT22], it is shown that binary polynomials with small symmetric expansion must be of certain special forms (strongly additive or strongly multiplicative, as we will introduce). The main technical lemma of this paper (4.19) then says that in locally modular reducts, all quasi-functions have small symmetric expansions. Combining with [JRT22], we then get that all binary polynomials in locally modular reducts must be of those special forms. Some legwork with polynomials is then carried out to complete the proof.

4.2 The Tools: Quasi-function and the SSE

We work in a fixed infinite model $M \models T$ of a complete T with elimination of imaginaries.

Definition 4.1. A definable function $f : X \to Y$ is *almost finite-to-one* if the union of infinite fibres of f has smaller Morley rank than X.

This definition makes sense because the union of infinite fibres is a definable set (since RM is definable in strongly minimal theories).

The definition of quasi-function below generalizes the essence of binary polynomial functions with respect to the Morley rank.

Definition 4.2. Let D, E be strongly minimal sets. A definable set $X \subseteq D^2 \times E$ is a *quasi-function* if $\mathrm{RM}(X) = 2$, $\mathrm{dM}(X) = 1$, and the projections $X \to D^2$ and $X \to D \times E$ (for both copies of D) are almost finite-to-one. A quasi-function X from D^2 to E is denoted $X : D^2 \dashrightarrow E$.

Lemma 4.3 (Quasi-functions generalize binary polynomials)**.** *Let $P(x, y) \in \mathbb{C}[x, y] - \mathbb{C}[x] - \mathbb{C}[y]$. Then, the graph of P is a quasi-function $\mathbb{C}^2 \dashrightarrow \mathbb{C}$ in the structure $(\mathbb{C}, +, \times)$.*

Proof. The graph of P clearly has Morley rank 2 and degree 1 because it projects bijectively to $\mathbb{C}^2$. The projection $\Gamma(P) \to \mathbb{C} \times \mathbb{C}$ to the second and third copies of $\mathbb{C}$ is almost finite-to-one: since $P \notin \mathbb{C}[y]$, for all but finitely many a, b, the equation $P(x, a) = b$ has only finitely many solutions. Similarly, the projection to the first and third copies of $\mathbb{C}$ is almost finite-to-one as $P \notin \mathbb{C}[x]$. □

The same proof also shows the graph of any definable polynomial in reducts of $\mathbb{C}$ is a quasi-function.

The generality of the quasi-function makes it a coarse-grained tool that allows passing through various natural operations associated with Morley rank.

Definition 4.4. Given two definable sets X and Y, we say

- X is *large* in Y if $\mathrm{RM}(Y - X) < \mathrm{RM}(Y)$;
- X is *small* in Y if $\mathrm{RM}(X \cap Y) < \mathrm{RM}(Y)$;
- X and Y are *almost equal* if X is large in Y and Y is large in X.

It should be clear that almost equality is an equivalence relation on definable sets. Moreover, two strongly minimal sets are almost equal if and only if they have finite symmetric difference. Finally, almost equality preserve Morley rank and degree.

Proposition 4.5 (Quasi-functions are defined modulo almost equality)**.** *Let D_1, D_2, E_1, E_2 be strongly minimal sets where D_1 a.e D_2 and E_1 a.e E_2.*

1. *If $X_1 \subseteq (D_1)^2 \times E_1$ and $X_2 \subseteq (D_2)^2 \times E_2$ are a.e, then X_1 is a quasi-function if and only if X_2 is a quasi-function.*

2. *If $X_1 : (D_1)^2 \dashrightarrow E_1$, then there is $X_2 : (D_2)^2 \dashrightarrow E_2$ such that X_2 a.e X_1.*

Proof. 1. Suppose X_1 is a quasi-function, we prove that X_2 is also a quasi-function.

Indeed, because Morley rank and degree are preserved by almost equality, $\mathrm{RM}(X_2) = 2$ and $\mathrm{dM}(X_2) = 1$. We just need to check that the projections are almost finite-to-one.

Write $X_i = Y_i \cup Z_i$ where Y_i is the union of infinite fibres of the projection $X_i \to (D_i)^2$, and Z_i is the union of finite fibres. We need to show Y_2 is small in X_i (i.e $\mathrm{RM}(Y_2) \leq 1$). Because Y_2 is the union of infinite fibres and each infinite fibre has $\mathrm{RM} = 1$, this amounts to showing that Y_2 is the union of finitely many infinite fibres.

Suppose the fibre of X_2 over $(d, d') \in (D_2)^2$ is infinite. If $(d, d') \notin (D_1)^2$ or if the fibre of X_1 over (d, d') is finite. Then this contributes a strongly minimal set towards $X_2 - X_1$. Because X_2 is almost equal to X_1, we must have $\mathrm{RM}(X_2 - X_1) \leq 1$. So there can only be finitely many such (d, d').

Similarly, the other two projections are also almost finite-to-one. This shows X_2 is a quasi-function.

2. Consider $X_2 = X_1 \cap ((D_2)^2 \times E_2)$. Equivalently, we remove $(D_1 - D_2) \times D_1 \times E_1$, $D_1 \times (D_1 - D_2) \times E_1$, and $D_1 \times D_1 \times (E_1 - E_2)$ from X_1 to obtain X_2. To see that X_2 a.e X_1, we just need to show that each of three sets removed have small intersection with X_1. We will show that $(D_1 - D_2) \times D_1 \times E_1$ has small intersection (i.e smaller RM than X_1) with X_1, and the other two follow analogously.

Write $X_1 = Y_1 \cup Z_1$ as above. Because X_1 is a quasi-function, the projection $X_1 \to (D_1)^2$ is almost finite-to-one. In particular, there are only a finite number of tuples $(d, d') \in (D_1 - D_2) \times D_1$ such that X_1 has infinite fibre over (d, d'). Then,

$$\mathrm{RM}(Y_1 \cap (D_1 - D_2) \times D_1 \times E_1) < 2 = \mathrm{RM}(X_1).$$

Because Z_1 projects finite-to-one to $(D_1 - D_2) \times D_1$ and finite-to-one maps do not decrease Morley rank, we also have

$$\mathrm{RM}(Z_1 \cap (D_1 - D_2) \times D_1 \times E_1) \leq \mathrm{RM}((D_1 - D_2) \times D_1) = 1 < \mathrm{RM}(X_1).$$

Combining, we get,

$$\mathrm{RM}(X_1 \cap (D_1 - D_2) \times D_1 \times E_1) < \mathrm{RM}(X_1).$$

Finally, because X_2 is a subset of X_1, it automatically has almost finite-to-one projections.

□

Proposition 4.5 shows that the notion of quasi-function is robust against small changes in the domain and codomain. If we treat almost equality as an equivalence relation on definable sets, then 4.5 says it makes sense to define a "quasi-function" from $[D^2]_{\text{a.e}}$ to $[E]_{\text{a.e}}$ as $[X]_{\text{a.e}}$ where X is a quasi-function from D^2 to E.

As promised, the next proposition shows that being a quasi-function is a property preserved under taking images and preimages of finite-to-one maps.

Proposition 4.6 (quasi-functions push forward and pull back)**.** *Let $f : D_1 \to D_2$ and $g : E_1 \to E_2$ be surjective finite-to-one definable functions, where D_i and E_i are strongly minimal. For $X_1 \subseteq (D_1)^2 \times E_1$, define $(f, f, g)(X_1) = \{(f(d), f(d'), g(e)) : (d, d', e) \in X_1\}$. Then*

1. *If $X_1 \subseteq (D_1)^2 \times E_1$ and $X_2 \subseteq (D_2)^2 \times E_2$ are such that $\mathrm{dM}(X) = 1$ and $X_2 = (f, f, g)(X_1)$, then X_1 is a quasi-function if and only if X_2 is a quasi-function.*

2. *For any $X_2 : (D_2)^2 \dashrightarrow E_2$, there is $X_1 : (D_1)^2 \dashrightarrow E_1$ such that $(f, f, g)(X_1) = X_2$.*

Proof. 1. Note that (f, f, g) is finite-to-one because f and g are finite-to-one. Therefore, $\mathrm{RM}(X_1) = \mathrm{RM}(X_2)$. Because (f, f, g) is surjective onto X_2, we see that $\mathrm{dM}(X_2) \leq \mathrm{dM}(X_1) = 1$, so $\mathrm{dM}(X_2) = 1$. So $(\mathrm{RM}, \mathrm{dM})(X_1) = (\mathrm{RM}, \mathrm{dM})(X_2)$.

It remains to look at the projections. By symmetry, we only look at the projections $X_i \to (D_i)^2$.

Write $X_i = Y_i \cup Z_i$ where Y_i is the union of infinite fibres of the projection $\pi_i : X_i \to (D_i)^2$ and Z_i is the union of finite fibres. We need to show Y_1 is small in X_1 if and only if Y_2 is small in X_2. Say $(d, d') \in (D_2)^2$ has an infinite fibre $(\pi_2)^{-1}(d, d') \subseteq Y_2$ over it. Then, because f, g are finite-to-one, there exists at least one and at most finitely many fibres in Y_1 whose image under (f, f, g) lies in $(\pi_2)^{-1}(d, d')$. This means Y_1 consists of finitely many fibres if and only if Y_2 consists of finitely many fibres. Each infinite fibre has Morley rank 1, so Y_i is small if and only if it contains only finitely many fibres. Thus, Y_1 is small if and only if Y_2 is small.

2. Let X_1' be a degree one component of the preimage of X_2 under (f, f, g). Then, $(f, f, g)(X_1')$ a.e. X_2 for degree reasons. Now add the preimage of $X_2 - (f, f, g)(X_1')$ under (f, f, g) to X_1' to obtain X_1 such that $(f, f, g)(X_1) = X_2$. Note that $\mathrm{dM}(X_1) = 1$. Therefore, we can apply part 1 to get that X_1 is a quasi-function.

□

Definition 4.7. Let D and E be strongly minimal sets and $X : D^2 \dashrightarrow E$ be a quasi-function. We say X has *small symmetric expansion* (SSE) if for all $\varepsilon > 0$, there

is arbitrarily large N and $A \subseteq D$ with $|A| = N$ such that $|X(A,A)| < N^{1+\varepsilon}$. We say X has *universal small symmetric expansion* (USSE) if every quasi-function from $D^2 \to D$ has SSE. Here, $X(A,A)$ denotes the set $\{e \in E : \exists a, a' \in A, (a, a', e) \in X\}$ and "arbitrarily large" means for every $N_0 \in \mathbb{N}$, there is $N > N_0$ (as opposed to sufficiently large).

Alternatively, we can view having an SSE as requiring the existence of a sequence as explicit witness.

Lemma 4.8. *$X : D^2 \dashrightarrow E$ has SSE if and only if there is a sequence $(A_k)_{k<\omega}$ with $A_k \subseteq D$ and $|A_k| \to \infty$ such that*

$$\limsup_{k\to\infty} \frac{\log |X(A_k, A_k)|}{\log |A_k|} \leq 1. \tag{4.2.1}$$

Proof. Notice that $|X(A,A)| < |A|^{1+\varepsilon}$ is equivalent to $\frac{\log |X(A,A)|}{\log |A|} < 1 + \varepsilon$, so this follows from a routine argument. □

Given that it makes sense to define quasi-function on strongly minimal sets modulo almost equality, it is a natural question to ask whether having SSE is preserved by almost equality. It is intuitive that if two quasi-functions are almost equal, then they behave similarly on large sets of inputs. Indeed, the following proposition confirms our intuition.

Proposition 4.9 (SSE is preserved by a.e)**.** *Let $X_1 : (D_1)^2 \dashrightarrow E_1$ and $X_2 : (D_2)^2 \dashrightarrow E_2$ be quasi-functions with X_1 a.e X_2, where D_i and E_i are as in proposition 4.5. Then, X_1 has SSE if and only if X_2 has SSE.*

Proof. Let $(A_k)_{k<\omega}$ witness X_1 having SSE. We will remove a bounded number of elements (say $\leq c$) from A_k to obtain a sequence $(A'_k)_{k<\omega}$ with $|X_2(A'_k, A'_k)| \leq |X_1(A_k, A_k)| + l|A_k|$ for some constant l. Indeed, we can see that this is enough to show (A'_k) witnesses X_2 having SSE as follows. For any $\varepsilon > 0$, we have

$$\frac{\log |X_2(A'_k, A'_k)|}{\log |A'_k|} \leq \frac{|X_1(A_k, A_k)| + l|A_k|}{\log(|A_k| - c)} \leq \frac{\log((l+1)|A_k|^{1+\varepsilon})}{\log(|A_k| - c)},$$

for large enough k. Therefore,

$$\limsup_{k\to\infty} \frac{\log |X_2(A'_k, A'_k)|}{\log |A'_k|} \leq \limsup_{k\to\infty} \frac{\log((l+1)|A_k|^{1+\varepsilon})}{\log(|A_k| - c)} \leq \frac{(1+\varepsilon)\log |A_k|}{\log |A_k|} = 1 + \varepsilon.$$

Taking $\varepsilon \to 0$, we see that $(A'_k)_{k<\omega}$ is a witness.

To produce A'_k, we first remove points $d \in A_k$ that are not in D_2. Because D_1 a.e D_2, this removes only a bounded number of points. Then, we remove points $d \in A_k$ such that $|(X_2 - X_1)(d, -)|$ is infinite (i.e points d such that there are infinitely many $(d', e) \in D_2 \times E_2$ with $(d, d', e) \in X_2 - X_1$). Because X_1 a.e X_2, $X_2 - X_1$ has RM ≤ 1,

and so there are only finitely many such points, bounded independent of k. Therefore, we know that $|A'_k| \geq |A_k| - d$ for some constant d.

The following is easily deduced from strong minimality of each component of X.

Fact 4.10. *If* $\mathrm{RM}(X) = 1$ *and* $\sim$ *is a definable equivalence relation on* X*, then the finite* $\sim$*-classes are bounded in size.*

By construction, $|(X_2 - X_1)(d, -)|$ is finite for all $d \in A'_k$. We know that $\mathrm{RM}(X_2 - X_1) = 1$. Define an equivalence relation on $X_2 - X_1$ where two tuples are equivalent if they have the same first coordinate, then $|(X_2 - X_1)(d, -)|$ is exactly the size of the equivalence class where the first coordinate is d. By the claim, we have a bound $|(X_2 - X_1)(d, -)| \leq l$ that holds for all $d \in A'_k$. Therefore, $|(X_2 - X_1)(A'_k, A'_k)| \leq l|A'_k| \leq l|A_k|$, so $|X_2(A'_k, A'_k)| \leq |X_1(A_k, A_k)| + l|A_k|$. This completes the proof. $\square$

Definition 4.11. Let $X : D^2 \dashrightarrow E$ be a quasi-function and $\sim$ be an equivalence relation on D with finite classes. We say X has *SSE respecting* D if X has a witness $(A_k)_{k<\omega}$ to SSE where each A_k is a union of $\sim$-classes.

Proposition 4.12 (SSE pull back and push forward)**.** *Let* $f : D_1 \to D_2$ *and* $g : E_1 \to E_2$ *be surjective finite-to-one definable functions, where* D_i *and* E_i *are strongly minimal. Suppose* $X_1 : (D_1)^2 \dashrightarrow E_1$ *and* $X_2 : (D_2)^2 \dashrightarrow E_2$ *be such that* $X_2 = (f, f, g)(X_1)$*. Let* $x \sim y$ *be the equivalence relation given by* $f(x) \sim f(y)$*. Then,* X_1 *has SSE respecting* $\sim$ *if and only if* X_2 *has SSE.*

Proof. Let $(A_k)_{k<\omega}$ be a sequence of finite subsets of D_1 where each A_k is a union of $\sim$-classes. Note that because A_k is a union of $\sim$-classes, $f^{-1}(f(A_k)) = A_k$. It is enough to show that (A_k) witnesses X_1 having SSE respecting $\sim$ if and only if $(f(A_k))_{k<\omega}$ witnesses X_1 having SSE. This is because any witness to X_2 having SSE must be of the from $(f(A_k))_{k<\omega}$ for some A_k union of $\sim$-classes.

First, we claim that A_k and $f(A_k)$ have the same asymptotic rate of growth. Formally, this means there are positive constants l_1, l_2 such that for all sufficiently large k, we have $l_1|A_k| \leq |f(A_k)| \leq l_2|A_k|$. Indeed, because f is finite-to-one and finite classes on strongly minimal sets are bounded in size, all $\sim$-classes have size smaller than some l'_1. Therefore, $l'_1|f(A_k)| \geq |A_k|$. Taking $l_1 = 1/l'_1$ and $l_2 = 1$ establishes this claim. Note that we only used the fact that f is finite-to-one on a strongly minimal set in this argument.

Second, we claim that $X_1(A_k, A_k)$ and $X_2(f(A_k), f(A_k))$ also have the same asymptotic rate of growth. Indeed, we have

$$\begin{aligned}
e_2 \in X_2(f(A_k), f(A_k)) &\iff \exists d, d' \in A_k, (f(d), f(d'), e_2) \in X_2 \\
&\iff \exists d, d' \in A_k, e_1 \in E_1, (d, d', e_1) \in X_1 \wedge g(e_1) = e_2 \\
&\qquad \text{(because } A_k \text{ is a union of } \sim\text{-classes)} \\
&\iff \exists e_1 \in X_1(A_k, A_k), e_2 = g(e_1) \\
&\iff e_2 \in g(X_1(A_k, A_k)).
\end{aligned}$$

Therefore, $g(X_1(A_k, A_k)) = X_2(f(A_k), f(A_k))$ and we can apply the same argument as in the previous claim, this time using the fact that g is finite-to-one on a strongly minimal set, to establish this claim.

With the first claim, we see that $|A_k|$ and $|f(A_k)|$ differ by only a constant. The second claim shows the same for $\log|X_1(A_k, A_k)|$ and $\log|X_2(f(A_k), f(A_k))|$. So $\limsup \frac{\log|X_1(A_k,A_k)|}{\log|A_k|} \leq 1$ if and only if $\limsup \frac{\log|X_2(f(A_k),f(A_k))|}{\log|f(A_k)|} \leq 1$. □

We mention some easy consequences that will be used throughout.

Corollary 4.13 (USSE is preserved by almost equality)**.** *Let D_1, D_2 be strongly minimal sets. If D_1 a.e. D_2, then D_1 has USSE if and only if D_2 has USSE.*

Proof. This is direct by 4.9. □

Corollary 4.14 (SSE respecting an finite equivalence relation is preserved by almost equality)**.** *Let D, E be strongly minimal and let $\sim$ be a definable equivalence relation on D with finite classes. If $X_1, X_2 : D^2 \dashrightarrow E$ are almost equal, then X_1 has SSE respecting $\sim$ if and only if X_2 has SSE respecting $\sim$.*

Proof. Consider the projection $\pi : D \to D/\sim$. Because X_1 and X_2 are almost equal, $(f, f, \mathrm{id})(X_1)$ almost equals $(f, f, \mathrm{id})(X_2)$. By 4.13, $(f, f, \mathrm{id})(X_1)$ has SSE if and only if $(f, f, \mathrm{id})(X_2)$. Apply 4.12 to finish. □

Corollary 4.15 (USSE pull back)**.** *Let $f : D_1 \to D_2$ be a definable surjection between strongly minimal sets. Then, D_2 has USSE implies D_1 has USSE.*

Proof. Given any quasi function $(D_1)^2 \dashrightarrow D_1$, use 4.6 to push to a quasi-function $(D_2)^2 \dashrightarrow D_2$, apply USSE of D_2, and use 4.12 to pull back. □

Lemma 4.16. *Let R be an integral domain of characteristic zero, $\sigma, \tau \in R$, and $H \subseteq R$ be a finite multiplicative subgroup of $R^\times$ equipped with action on R by scaling. Then, we can find increasingly large H-invariant sets $S_k \subseteq R$ with $0 \in S_k$ and*

$$\limsup_{k\to\infty} \frac{\log|\sigma \cdot S_k + \tau \cdot S_k|}{\log|S_k|} \leq 1. \tag{4.2.2}$$

Proof. Note that if $(S_k)_{k<\omega}$ is a sequence of finite subsets of the field of fractions of R satisfying (4.2.2 and $(c_k)_{k<\omega}$ is a sequence of nonzero elements of R, then $(c_k \cdot S_k)_{k<\omega}$ always satisfies 4.2.2. So we may pass to the field of fractions and assume R is a field.

Similarly, $(S_k)_{k<\omega}$ witnesses 4.2.2 for σ and τ if and only it witnesses 4.2.2 for σ/r and τ/r for any $r \in R^\times$. So, by rescaling, we may assume $\sigma \in \mathbb{Z}$. We will first prove the case where τ is algebraic over $\mathbb{Q}$.

If $\tau = 0$, there is nothing to prove. So assume τ has degree $d > 0$ over $\mathbb{Q}$. Then, we can write τ as the quotient of an algebraic integer by an integer. Indeed, for any integer polynomial p with $p(\tau) = 0$ and $a_{d'}$ being the coefficient of the highest degree

term in p, we have that $a_{d'}\tau$ is an algebraic integer. Rescaling σ, τ by an integer, we may assume that τ is an algebraic integer. Let

$$B_k = \{a_0 + a_1\tau + \ldots + a_{d-1}\tau^{d-1} : a_i \text{ is an integer in } [-k, k]\}$$

and

$$S_k = \left\{ \sum_{h \in H} h \cdot a_h : a_h \in B_k \right\}.$$

It is clear that $0 \in S_k$ and S_k is H-invariant.

We claim that $\sigma \cdot S_k + \tau \cdot S_k \subseteq S_{Nk}$ for some constant N independent of k. It suffices to show this for B_k. Let $N_0 \in \mathbb{Z}$ bound the absolute value of the coefficients of the minimal polynomial of τ. Then, for any $a_0 + a_1\tau + \ldots + a_{d-1}\tau^{d-1} \in B_k$, we get that $\tau(a_0 + a_1\tau + \ldots + a_{d-1}\tau^{d-1}) \in B_{(N_0+1)k}$. So if $a, b \in B_k$, then $\sigma \cdot a + \tau \cdot b \in B_{(N_0+|\sigma|+1)k}$, establishing our claim with $N = N_0 + |\sigma| + 1$.

Take any $a_0 + \ldots + a_{d-1}\tau^{d-1} \in B_{Nk}$, we can write $a_i = b_i \cdot k + c_i$ where $b_i \in [-N, N]$ and $c_i \in [-k, k]$. So

$$\sum_i a_i\tau^i = k \sum_i b_i\tau^i + \sum_i c_i\tau^i \in k \cdot B_N + B_k.$$

Therefore, $\sigma \cdot S_k + \tau \cdot S_k \subseteq S_{Nk} \subseteq k \cdot S_N + S_k$ and subsequently

$$|\sigma \cdot S_k + \tau \cdot S_k| \leq |S_N||S_k|.$$

It follows that $\limsup_k \frac{\log |\sigma \cdot S_k + \tau \cdot S_k|}{\log |S_k|} \leq 1$, proving the algebraic case.

Now suppose τ is transcendental over $\mathbb{Q}$. For any $U \subseteq Q(H)$ and positive integer d, we let $B_d(U) = \{a_0 + \ldots + a_{d-1}\tau^{d-1} : a_i \in U\}$. Because H is a finite group, we know that each element of H is algebraic over $\mathbb{Q}$, so τ is transcendental over $\mathbb{Q}(H)$. It follows that for any $U \subseteq Q(H)$, $|B_d(U)| = |U|^d$. Now, observe that if $0 \in U \subseteq Q(H)$, then $\sigma \cdot B_d(U) + \tau \cdot B_d(U) \subseteq B_{d+1}(\sigma \cdot U + U)$. Indeed, given any $a_0 + \ldots + a_{d-1}\tau^{d-1}, b_0 + \ldots + b_{d-1}\tau^{d-1} \in B_d(U)$, we have

$$\sigma \left(\sum_i a_i\tau^i \right) + \tau \left(\sum_i b_i\tau^i \right) = \sigma a_0 + (\sigma a_1 + b_0)\tau + \ldots + (\sigma a_{n-1} + b_{n-2})\tau^{d-1} + b_{n-1}\tau^n,$$

which is in $B_{d+1}(\sigma \cdot U + U)$.

This suggests we apply the algebraic case to $R' = \mathbb{Q}(H)$ with σ and $\tau' = 1$: we get a sequence of increasingly large H-invariant finite sets $(U_k)_{k<\omega}$ containing 0 with $\limsup_k \frac{\log |\sigma \cdot U_k + U_k|}{\log |U_k|} \leq 1$. Let us set $S_k = B_k(U_k)$. This works:

- S_k is H-invariant because U_k is;
- S_k contains 0 because U_k does;

- and finally,

$$\limsup_k \frac{\log|\sigma \cdot S_k + \tau \cdot S_k|}{\log|S_k|} \leq \limsup_k \frac{k+1}{k} \frac{\log|\sigma \cdot U_k + U_k|}{\log|S_k|} \leq 1.$$

□

4.3 Nontrivial Locally Modular Polynomial Reducts of $(\mathbb{C}, +, \times)$

We now apply the tools above to prove Theorem 1.2. Fix a nontrivial locally modular polynomial reduct $M = (\mathbb{C}, \mathcal{P})$ of $(\mathbb{C}, +, \times)$.

The following fundamental results will be of central importance.

Fact 4.17 (classification of definable sets in locally modular groups [HP87])**.** *Suppose M is a strongly minimal, nontrivial, and locally modular structure. Then, there is a definable, strongly minimal, abelian group $(G, \ldots)$ in M^{eq} in definable finite correspondence with M. Moreover, for any such G, every definable subset of every G^n is a boolean combination of cosets of definable subgroups of G^n.*

This classification is the key tool throughout this chapter. It is powerful as the conclusion asserts that *all* definable relations come from a group operation.

Note that by 3.16, the G in the classification must also be locally modular.

The following result applies 4.17 to classify all degree one definable sets in G up to almost equality.

Lemma 4.18 (degree one sets a.e degree one cosets in locally modular groups)**.** *Suppose G is a locally modular strongly minimal group definable in M. Suppose $X \subseteq G^n$ has $\mathrm{dM}(X) = 1$. Then, X a.e C where C is a coset of a subgroup of G^n.*

Proof. By 4.17, we can write X as a boolean combination of cosets of G^n. Putting the boolean combination into DNF and using the fact that RM of a union is the max of the RM's, we see that X is almost equal to the intersection of cosets and complements of cosets in G^n. By elementary group theory, the intersection of cosets is a coset, so we may assume X a.e $Y = C_0 - C_1 - C_2 - \ldots - C_k$ where C_i are cosets. Without loss of generality, we may assume $C_i \subseteq C_0$ for all $i \geq 1$. Indeed, if $\mathrm{RM}(C_i) < \mathrm{RM}(C_0)$, then the $-C_i$ in the expression for Y does not change the fact that X almost equals Y. Discarding all C_i such that $\mathrm{RM}(C_i) < \mathrm{RM}(C_0)$, we may assume $\mathrm{RM}(C_i) = \mathrm{RM}(C_0)$ for all i.

If C_0 is a coset of some $H \leq G^n$, then, for each $i \geq 1$, C_i is the coset of some subgroup H_i of H. Because C_0 has finite degree, finitely many cosets of H_i covers C_0. Thus at least one of the cosets of H_i has an intersection with X of $\mathrm{RM} \geq \mathrm{RM}(X)$. Let C_i' be such a coset. But because $\mathrm{dM}(X) = 1$, C_i' is the unique coset of H_i

whose intersection with X is large in X. Consider the coset $C = \bigcap_{i\geq 1} C_i'$. Clearly, $\bigcap_{i\geq 1} C_i' \subseteq Y$.

Moreover,

$$\mathrm{RM}\left(\bigcap_{i\geq 1} C_i' \cap X\right) = \mathrm{RM}\left(\bigcap_{i\geq 1}(C_i' \cap X)\right) = \mathrm{RM}(X).$$

The second equality follows because each $C_i' \cap X$ almost equals X, and therefore the entire intersection almost equals X.

To summarize, C is a coset that is large in Y, and has equal Morley rank as X. Because Y almost equals X, we get C almost equals X. □

We may add a countably infinite set of constant symbols to the language of M without changing the fact that it is a nontrivial locally modular polynomial reduct of $(\mathbb{C}, +, \times)$ (local modularity is preserved by 3.11, nontriviality is preserved by saturation of M). Doing so also does not change the class of definable sets. Therefore, we may assume without loss of generality that $\mathrm{acl}(\varnothing)$ is infinite. Together with the fact that M is strongly minimal, we get that M has weak elimination of imaginaries (see 8.4.11 of [TZ12]). Using a compactness argument, we see that every strongly minimal set in M^{eq} is in definable finite correspondence with M.

Proposition 4.19. *Suppose that every strongly minimal group in M^{eq} is divisible with a commutative ring of definable endomorphisms, then every strongly minimal set in M^{eq} has USSE.*

Proof. By the comment before this proposition, we have that every strongly minimal set in M^{eq} is in finite correspondence with $\mathbb{C}$. Therefore, every two strongly minimal set in M^{eq} are in finite correspondence.

By Fact 4.17, there is a strongly minimal group in M^{eq}. Because locally modularity is preserved by finite correspondence (3.16), we see that G is also locally modular.

Let D be any strongly minimal set in M^{eq}, so D is in finite correspondence with G. We will show that D has USSE after several reductions.

First, we claim that we can assume D is of the form $D'/\sim$ where $D' \subseteq G^n$ is a strongly minimal set and $\sim$ is a definable equivalence relation whose equivalence classes are finite.

Let $C \subseteq D \times G$ be the finite correspondence between D and G. Because the projection of C to D is almost equal to D and USSE is preserved by almost equality (4.13), we can replace D with the projection of C to D. Letting $C(d)$ denote the set of $g \in G$ such that $(d, g) \in C$, we view C as a multi- (but finite) valued, finite-to-one function from D to G. Note that for any $d \in D$, $n \in \mathbb{N}$, "$|C(d)| = n$" is a first order property of d. By strong minimality of D, we know that there is a positive integer n so that $|C(d)| = n$ for cofinitely many $d \in D$. Define $F : D \to G^{(n)}$ (here and throughout, $G^{(n)}$ denotes the nth symmetric product of G, or equivalently the set of

n-element subsets of G) as follows. If $|C(d)| = n$, let $F(d) = C(d)$; otherwise, set $F(d)$ to some arbitrary value in $G^{(n)}$. It is clear that F is a definable, finite-to-one function on D. In particular, $F(D)$ is strongly minimal.

Because USSE pull back (4.15), $F(D)$ having USSE would imply that D has USSE. Consider the preimage of $F(D)$ under the projection $G^n \to G^{(n)}$. Because the projection is finite-to-one, the preimage is also of Morley rank 1. Let D' be a degree 1 component of the preimage. Then, $F(D)$ almost equals $D'/\sim$ where $\sim$ is the equivalence relation given by the projection. Because USSE is preserved by almost equality (4.13), we may replace D with $D'/\sim$, achieving our first reduction.

By locally modularity of G, we see from Lemma 4.18 that D' is almost equal to a definable coset C of a subgroup of G^n. So $D'/\sim$ has USSE if and only if $C/\sim$ has USSE (technically, we need to modify $\sim$ at finitely many points so that it is an equivalence relation on C). We have thus reduced to the case where $D = C/\sim$.

Say $C = c + H$ for some strongly minimal subgroup H of G^n. Define $\sim'$ on G^n by $h \sim' k$ if and only if $h + c \sim k + c$, then it's clear that there is a definable bijection from $H/\sim'$ to $C/\sim$ (by sending $[h]_{\sim'}$ to $[h + c]_\sim$). Therefore, we may assume that $D = H/\sim$ where H is a strongly minimal subgroup in G^n and $\sim$-classes are finite.

Furthermore, note that H is also strongly minimal and locally modular.

Therefore, we might as well assume H is G so that D has the form $G/\sim$ where $\sim$-classes are finite.

Let us reduce to the case where $\sim$-classes are "small" in the further sense that translations by elements of G do not preserve $\sim$-classes. More precisely, let $L \leq G$ be the group of $g \in G$ such that $\{h \in G : h + g \sim h\}$ is cofinite. We call the elements of L $\sim$-translations. We wish to reduce to the case when L is the trivial subgroup.

Clearly, L is a finite subgroup of G: for any distinct $g_1, \ldots, g_n \in L$, we can find $h \in G$ so that $h \sim h + g_1 \sim \ldots \sim h + g_n$ (this is because the intersection of finitely many cofinite set is nonempty); thus the finiteness of L follows from the finiteness of $\sim$-classes. It is also clear from the definition that all but finitely many $\sim$-classes are L-invariant.

By preservation of USSE under almost equality (4.13), $G/\sim$ has USSE if and only if the set of L-invariant classes has USSE, which can be viewed as a quotient of G/L precisely as follows. Define an equivalence relation $\sim'$ on G/L where $g + L \sim' h + L$ if and only if

- each of $g + L$ and $h + L$ intersects some $\sim$-class that is not L-invariant, or
- both are subsets of the same L-invariant $\sim$-class.

Consider the function sending each L-invariant $[g]_\sim$ to $[g+L]_{\sim'}$. This is a definable injection from the set of L-invariant $\sim$-classes to $(G/L)/\sim'$ with cofinite image (in fact, at most one point in $(G/L)/\sim'$ lies outside the image). Therefore, $G/\sim$ has USSE if and only if $(G/L)/\sim'$ has USSE. Furthermore, notice that G/L has no nontrivial $\sim'$-translations: $g + L \sim' g + h + L$ for cofinitely many cosets $g + L$ implies

$g \sim g + h$ for cofinitely many $g \in G$, so $h \in L$. Because local modularity is preserved by definable quotients with finite fibres (3.12), G/L satisfies the same conditions as G (local modularity, finite correspondence with M, strong minimality, every strong minimal group in $(G/L)^{eq}$ is divisible with commutative definable endomorphism ring). So we may assume $D = G/\sim$ where G has no $\sim$-translations.

Claim 4.19.1. *Let $X \subseteq G^2$ be a strongly minimal subset of the graph of $\sim$. Then there are a definable automorphism σ of G and $b \in G$ such that X is almost equal to the graph of $x \mapsto \sigma(x) + b$.*

Proof. By 4.18, X is almost equal to a coset C of a strongly minimal subgroup in G^2. The projection of C to either copy of G is a coset of G, so it is either finite or the entire G. Say the projection is finite. Then it has to be a singleton $\{g\}$ because C is strongly minimal. This contradicts the fact that $\sim$ has finite classes.

Therefore, we can assume that $C \to G$ is surjective for both copies of G. It follows from elementary group theory that the fibres of the projection (say, to the first copy of G) are cosets of the same subgroup $L \leq G$. Take any $0 \neq h \in L$. Because X is almost equal to C, we get that for cofinitely many $g \in G$, $g \sim g + h$. This is impossible. Therefore, the projection $C \to G$ is bijective for both copies of G, and so C is the graph of some $\sigma(x) + b$ as desired. □

Let H be the set of maps $x \mapsto \sigma(x) + b$ whose graph is equal to some strongly minimal subset of the graph of $\sim$. Then H is finite because the graph of $\sim$ has Morley rank 1 (it projects finite-to-one onto G). It's clear that it also forms a group under composition.

The union of graphs of maps in H is almost equal to the graph of $\sim$ and the action of H induces a definable equivalence relation on G. Therefore, we may assume that $\sim$ is given by the action of H on G (this can be justified in a similar way to how we reduced to G/L above).

Consider the group homomorphism that sends $(x \mapsto \sigma(x) + b) \in H$ to σ in the multiplicative group of the ring of definable endomorphisms of G, which is assumed to be commutative. This map is injective because each σ can only appear in one element of H. Indeed, if σ appear twice, then we would contradict the fact that G has no nontrivial $\sim$-translation. Therefore, H is isomorphic to a finite subgroup of the multiplicative group of an integral domain (a nonzero definable endomophisms of a strongly minimal group is always surjective, hence we have an integral domain). It follows that H is cyclic. Let $h = (x \mapsto \sigma(x) + b) \in H$ be a generator. Notice that h has a unique fixed point: clearly, σ is not the identity, so $\sigma - 1$ is nonzero and therefore surjective; say $(\sigma - 1)(b') = b$, then b' is the fixed point of h. By applying a translation, we can reduce to the case where 0 is the fixed point of h. This is formally justified by defining a bijection from $G/\sim$ to $G/\sim'$ in M^{eq} where $x \sim' y$ if and only if $x + b' \sim' y + b'$. After reduction, H is just a multiplicative subgroup of the definable endomorphism ring of G. In other words, D is the quotient of G by the action of a finite group of definable endomorphisms of G.

Let $X : D^2 \dashrightarrow D$ be a quasi-function. By 4.6, we can find $X' : G^2 \dashrightarrow G$ such that $\pi(X') = X$. We just need to show that X' has a SSE respecting the action of H. By 4.18, X' is almost equal to a coset of G^3. Because SSE respecting an equivalence relation is preserved under almost equality (4.14), we may assume X' is a coset. Consider the projection $X' \to G^2$ (the first two copies of G). Because the Morley rank of X' is 2 and the projection is almost finite-to-one, it follows that X' projects to a rank two coset of G^2. Because $\mathrm{RM}(G^2) = 2$ and $\mathrm{dM}(G^2) = 1$, the projection is surjective. From elementary group theory, we see that all fibres of the projection are finite cosets of a fixed finite subgroup of G, say, of order m. Apply the map $g \mapsto mg$ on the third coordinate of X', we can assume by 4.12 (with f the identity) that X' is the graph of a function $G^2 \to G$. In fact, X' must be the graph of some $(x, y) \mapsto \sigma(x) + \tau(y) + b$ for some $b \in G$ and definable endomorphisms σ and τ of G. By 4.12, we can assume $b = 0$.

Taking R as the ring of definable endomophisms of G, we can apply 4.16 to $\sigma, \tau \in R$ to get increasingly large H-invariant sets of endomorphisms $S_k \subseteq R$ with

$$\limsup_{k\to\infty} \frac{\log |\sigma \cdot S_k + \tau \cdot S_k|}{\log |S_k|} \leq 1.$$

Because nonzero endomorphisms of G have finite kernels (this is direct from G being strongly minimal), and saturation, we can find $x_0 \in G$ such that $|S_k(x_0)| = |S_k|$. Take $A_k = S_k(x_0)$. The fact that A_k is a small symmetric expansion of X' follows from $|\sigma \cdot S_k + \tau \cdot S_k| = |X'(A_k, A_k)|$. □

Fact 4.20 (Theorem 4.13, [Poi01])**.** *Every definable group in $\mathbb{C}^n$ is definably (in $(\mathbb{C}, +, \times)$) isomorphic to an definable algebraic group.*

Lemma 4.21. *Let $P(x, y) \in \mathbb{C}[x, y] - \mathbb{C}[x] - \mathbb{C}[y]$ be a polynomial definable in M. Then P has SSE.*

Proof. By Proposition 4.19, it suffices to show that if G is a definable strongly minimal group in M^{eq}, then G is divisible and its ring of definable endormophisms is commutative.

Since M is a reduct of $(\mathbb{C}, +, \times)$, G is also definable in $(\mathbb{C}, +, \times)$, and hence by 4.20 G is $(\mathbb{C}, +, \times)$-definably isomorphic to an algebraic group H.

As we have noted, G is in definable finite correspondence with $\mathbb{C}$. Hence H is in $(\mathbb{C}, +, \times)$-definable correspondence with $\mathbb{C}$. In particular, this means H is a one-dimensional algebraic group.

The identity component H_0 of H, is therefore a connected one-dimensional algebraic group over $\mathbb{C}$. It is well-know that the only possibilities for these are $(\mathbb{C}, +)$, $(\mathbb{C}^\times, \times)$, and elliptic curves. In particular, H is not a torsion group.

We claim that this forces G to be divisible. Fix $k > 0$ and consider the M-definable endomorphism $[k] : G \to G$ given by multiplication by k. We need to show that it is surjective. As G is strongly minimal, it suffices to show that the image $[k]G$

is not finite. But as G is connected (by strong minimality), $[k]G$ can only be the trivial subgroup if it is finite. This is not possible because G is not torsion. So G is divisible.

But this in turn implies that H is divisible. Since H/H_0 is finite (H/H_0 makes sense H_0 is a normal subgroup of H), and hence of finite order, divisibility forces $H = H_0$. That is, H itself is isomorphic to $(\mathbb{C}, +)$, $(\mathbb{C}^\times, \times)$, or an elliptic curve.

To summarize, we have show that if G is a strongly minimal group in M^{eq}, then G is divisible and it is $(\mathbb{C}, +, \times)$-definably isomorphic to an algebraic group H, where H is either $(\mathbb{C}, +)$, $(\mathbb{C}^\times, \times)$, or an elliptic curve. Now, the ring of M-definable endomorphisms $\mathrm{Hom}_M(G)$ of G can be viewed as a subring of $\mathrm{Hom}_{(\mathbb{C},+,\times)}(H)$, which is well-known to be commutative. Therefore $\mathrm{Hom}_M(G)$ is commutative, as desired. □

Most of the difficult work is completed at this point. We apply the results to show that any locally modular reduct must be interdefinable with the twisted multiplication reduct or a vector space reduct. The exposition follows the original paper quite closely.

Definition 4.22. Let $P(x, y) \in \mathbb{C}[x, y] - \mathbb{C}[x] - \mathbb{C}[y]$.

- P is *weakly additive* if there are nonconstant unary polynomials f, u, v such that $P(x, y) = f(u(x) + v(y))$.
- P is *strongly additive* if there are nonconstant unary polynomials f, u and constants a, b such that $P(x, y) = f(au(x) + bu(y))$.
- P is *weakly multiplicative* if there are nonconstant unary polynomials f, u, v such that $P(x, y) = f(u(x)v(y))$.
- P is *strongly multiplicative* if there are nonconstant unary polynomials f, u and $m, n \geq 0$ such that $P(x, y) = f(u^m(x)u^n(y))$.

It is clear that strongly additive/multiplicative implies weakly additive/multiplcative.

A key ingredient is that SSE implies strong additivity or strong multiplicativity, as was shown earlier in [JRT22].

Fact 4.23 (Theorem 1.1, [JRT22])**.** *If $P(x, y) \in \mathbb{C}[x, y] - \mathbb{C}[x] - \mathbb{C}[y]$ has SSE, then P is strongly additive or strongly multiplicative.*

Together with Lemma 4.21, we get the following.

Corollary 4.24. *Let $P(x, y) \in \mathbb{C}[x, y] - \mathbb{C}[x] - \mathbb{C}[y]$ be a polynomial definable in M. Then P is strongly additive or strongly multiplicative.*

Lemma 4.25. *Let $P(x, y) \in \mathbb{C}[x, y] - \mathbb{C}[x] - \mathbb{C}[y]$ be a polynomial definable in M.*

- *If P is weakly additive, then P is nonconstant on every horizontal and vertical line.*

- *If P is weakly multiplcative, P is constant on at least one horizontal and one vertical line.*
- *P is not both weakly additive and weakly multiplicative.*

Proof. If $P(x,y) = f(u(x)+v(y))$. Then, as f and u are nonconstant, $P(x,y)$ must not be constant on any horizontal line. Similarly for every vertical line.

If $P(x,y) = f(u(x)v(y))$. Choose y_0 such that $v(y_0) = 0$. Then on the horizontal line $y = y_0$, $P(x,y)$ is constant. Similarly for some vertical line.

The last point then follows. □

Corollary 4.26. *Let $P(x,y) \in \mathbb{C}[x,y] - \mathbb{C}[x] - \mathbb{C}[y]$ be a polynomial definable in M.*

- *P is strongly additive if and only it is weakly additive; P is strongly multiplicative if and only if it is weakly additive.*
- *If P is weakly additive, then $\deg_x(P) = \deg_y(P)$.*
- *If $P(x,y) = f(u(x)v(y))$, then u and v have the same roots.*

Proof. The first point follows from the fact that P is strongly additive or strongly multiplcative (4.24) and that it cannot be both weakly additive and weakly multiplicative.

The rest follows from the first point. □

Proposition 4.27. *Let $P(x,y) \in \mathbb{C}[x,y] - \mathbb{C}[x] - \mathbb{C}[y]$ be a strongly additive polynomial definable in M. Then $P(x,y)$ is linear.*

Proof. Write $P(x,y) = f(au(x)+bu(y))$.

Claim 4.27.1. *Every unary polynomial g definable in M is linear.*

Proof. We may assume g is not constant. Define $Q(x,y) = P(x,g(y)) = f(au(x) + bv(g(y)))$, which is weakly additive. By Corollary 4.26, Q is strongly additive. Therefore, $\deg_x(Q) = \deg_y(Q)$, which implies $\deg(g) = 1$. □

Choose $y \in \mathbb{C}$ so that $u(y) = 0$. Then, by the claim, $g(x) = f(au(x))$ is linear. This implies f and u are linear, and so is P. □

Recall that the twisted multiplication is defined by $a \times_r b = (a-r)(b-r)+r$.

Definition 4.28. A *monomial twisted by r* is a polynomial of the form $a \times_r x_1 \times_r \ldots \times_r x_n$.

Lemma 4.29. *Suppose f and g are nonconstant unary polynomials such that f and $f \circ g$ have the same roots.*

- *If g is not linear, then f has only one root.*

- *If r is the unique root of f, then g is a monomial twisted by r.*

Proof. Let R be the set of roots of f. Then f and $f \circ g$ having the same roots amounts to saying $g(R) \subseteq R$ and $g^{-1}(R) \subseteq R$. Because g is surjective and $g^{-1}(R) \subseteq R$, we have $g(R) = R$. This implies $g^{-1}(R) \supseteq R$. Hence $g(R) = g^{-1}(R) = R$.

Let $r \in R$ and $s = g(r)$. Then r is the unique solution to $g(x) = s$. Therefore we can write $g(x) = a(x-r)^d + s$ where $d = \deg(g)$.

For the first point, suppose g is not linear. Then $d > 1$ and $g'(x) = ad(x-r)^{d-1}$ is not constant. Notice that r is the only root of g', which means g uniquely determines r. Therefore the only root of f is r.

For the second point, suppose r is the unique root of f. Then $s = r$, giving $g(x) = a(x-r)^d + r$. □

Corollary 4.30. *Let $P(x,y) \in \mathbb{C}[x,y] - \mathbb{C}[x] - \mathbb{C}[y]$ be a strongly multiplcative polynomial definable in M. Then $P(x,y)$ is a twisted monomial.*

Proof. Write $P(x,y) = f(u^m(x)u^n(y))$.

Claim 4.30.1. *u has a unique root r and every unary polynomial g definable in M is a monomial twisted by r.*

Proof. Define $Q(x,y) = P(x, g(y)) = f(u^m(x)u^n(g(y)))$. Then, by Corollary 4.24, Q is strongly multiplicative. Therefore, u^m and $(u \circ g)^n$ have the same roots. This implies u and $u \circ g$ have the same roots.

If g is nonlinear, then we can apply Lemma 4.29 to see that u has a unique root. If not, consider $g'(x) = Q(x,x)$. Then g' is nonlinear, definable in M. Our conclusion from the previous paragraph holds for g' (i.e u and $u \circ g'$ have the same roots). Lemma 4.29 implies u has a unique root r.

Apply 4.29 again, we see that g must be a monomial twisted by r. □

Pick y such that $u(y) = 1$. Then $f(u^m(x))$ is definable in M. By our claim, $f(u^m(x))$ is a monomial twisted by r. Write $f(u^m(x)) - r = a(x-r)^d$. Note that because r is the only root of u, we can write $u(x) = b(x-r)^e$ for some b, e. Therefore, we get $f(b^m(x-r)^{me}) - r = a(x-r)^d$. Write $z = x - r$, we get $f(b^m z^{me}) = az^d$. This is only possible if $f(x)$ is some monomial x^k. Therefore, we have

$$f(u^m(x)u^n(y)) = ((b(x-r)^e)^m(b(y-r)^e)^n)^k + r = b^{mk+nk}(x-r)^{me}(y-r)^{ne} + r,$$

which is a monomial twisted by r. □

Proposition 4.31. *1) If some polynomial in $\mathcal{P}$ is linear and depends on at least two variables, then M defines $+$.*

2) If all polynomials in $\mathcal{P}$ are linear, then M is interdefinable with the F-vector space structure $(\mathbb{C}, +, (\lambda\cdot)_{\lambda \in F})$ where F is the subfield of $\mathbb{C}$ generated by coefficients of nonconstant terms of polynomials in $\mathcal{P}$.

3) If some polynomial in $\mathcal{P}$ is a monomial twisted by r and depends on at least two variables, then M defines $\times_r$.

4) If there exists $r \in \mathbb{C}$ such that all polynomials in $\mathcal{P}$ are monomials twisted by r, then M is interdefinable with $(\mathbb{C}, \times_r)$.

Proof. 1) Suppose $P(x_1, \ldots, x_n) \in \mathcal{P}$ is linear and x_1, x_2 occur in P. We can write

$$P(x_1, \ldots, x_n) = b + \sum_k a_k x_k.$$

By setting all variables other than x_1 so that $\sum_{k>1} a_k x_k = -b$, we get the that $x \mapsto a_1 x$ is definable in M. Similarly, we get that $x \mapsto a_2 x$ is definable. Therefore the inverses $x \mapsto a_1^{-1} x$ and $x \mapsto a_2^{-1} x$ are definable. Setting all variables in P other than x_1 to 0, we get the that $x \mapsto a_1 x + b$ is definable. Precomposing with $x \mapsto a_1^{-1} x$, we get the map $x \mapsto x + b$, which means $x \mapsto x - b$ is definable. Setting all variables in P other than x_1, x_2 to 0, we get the map $x_1, x_2 \mapsto a_1 x_1 + a_2 x_2 + b$. Postcomposing with $x \mapsto x - b$ and precomposing with $x_1 \mapsto a_1^{-1} x_1$ and $x_2 \mapsto a_2^{-1} x_2$, we recover the addition $x, y \mapsto x + y$.

2) It is clear that F-vector space structure defines all polynomials in $\mathcal{P}$. The other direction follows from 1). Indeed, by nontriviality of M and the trivial part of Theorem 1.1, some polynomial in $\mathcal{P}$ depends on at least two variables. Applying 1), we get that $x, y \mapsto x + y$ is definable. Then, $x \mapsto 0$ and $x \mapsto -x$ are definable. It is then clear that if a is the coefficient of a nonconstant term of a polynomial in $\mathcal{P}$, then $x \mapsto ax$ is definable. If λ can be obtained by taking products and quotients of such a's, then $x \mapsto \lambda x$ is also definable. Hence the F-vector space structure is definable.

3) Suppose $P(x, y, \ldots) = a(x - r)^m (y - r)^n (\ldots) + r \in \mathcal{P}$. Setting all variables other than x, y appropriately so that $(\ldots) = a^{-1}$, we get

$$x, y \mapsto \underbrace{x \times_r \ldots \times_r x}_{m \text{ times}} \times_r \underbrace{y \times_r \ldots \times_r y}_{n \text{ times}}$$

is definable. Similarly, we get

$$x \mapsto \underbrace{x \times_r \ldots \times_r x}_{m \text{ times}} \text{ and } y \mapsto \underbrace{y \times_r \ldots \times_r y}_{n \text{ times}}.$$

Precomposing the inverses of these with the function above, we get $x, y \mapsto x \times_r y$.

4) Suppose M is interdefinable with $(\mathbb{C}, \times_r)$. Then it's clear that we can recover any monomial twisted by r. Note that by the trivial case of Theorem 1.1, there is a polynomial in $\mathcal{P}$ in at least two variables. The converse then follows from 3).

□

Theorem 4.32. *M is interdefinable with a vector space or a twisted multiplication.*

Proof. By nontriviality of M, there is some $P(x_1, \ldots, x_n) \in \mathcal{P}$ involving at least two variables. We can specialize all other variables so that P is a nonzero binary polynomial (say x_1, x_2 actually occur in $P(x_1, \ldots, x_n)$; take any $a_3, \ldots, a_n$ such that $P(x_1, x_2, a_3, \ldots, a_n) \not\equiv 0$).

By Corollary 4.24, Proposition 4.27, and Corollary 4.30, P must be linear or a twisted multiplication. By Proposition 4.31 1) and 3), we get that $+$ or $\times_r$ is definable in M. Let $\otimes$ denote this group operation.

Let $P' \in \mathcal{P}$. View P' as a function from $M^m \to M$. Then the graph of P' has Morley degree one and rank m in M. By Lemma 4.18, $\Gamma(P')$ almost equals a degree one coset $C \subseteq M^{m+1}$ of some subgroup of $(M^{m+1}, \otimes^{m+1})$. By almost equality with $\Gamma(P')$, the projection $C \to M^m$ is almost finite-to-one. But C is a coset, so the projection is in fact a bijection. It follows from elementary group theory that C is the graph of some $Q : (x_1, \ldots, x_m) \mapsto a \otimes \sigma_1(x_1) \otimes \ldots \otimes \sigma_m(x_m)$ for some $a \in M$ and $\otimes$-endomorphisms σ_i. If $\otimes$ is $+$, then Q is a linear polynomial; otherwise, Q is a monomial twisted by r. In any case, because $\Gamma(P')$ almost equals C, we get $\mathrm{RM}^{\mathbb{C}}(V(P' - Q)) \geq \mathrm{RM}^{M}(V(P' - Q)) = m$. Here, $V(\cdot)$ denotes the zero set. In $\mathbb{C}$, Morley rank corresponds to the dimension of $V(P' - Q)$ as a variety. Thus $P' = Q$.

This shows either all polynomials in $\mathcal{P}$ are linear, or all are monomials twisted by r. By 2) and 4) of Proposition 4.31, we see that M is interdefinable with some F-vector space or $(\mathbb{C}, \times_r)$. $\square$

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